Centro di Ricerche e Documentazione "Piero Sraffa"

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Centro Sraffa Working Papers
n. 66

June 2024

ISSN: 2284-2845
Centro Sraffa working papers
[online]

Visualizing Variations in the Analysis of the Choice of Technique

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Abstract

This article describes a diagram that depicts how the analysis of the choice of technique varies with perturbations of selected parameters in models of production of commodities. Fluke switch points partition the graph. Three examples are provided, of circulating capital with markup pricing, of fixed capital with structural economic dynamics, and of intensive rent with markup pricing.

Keywords: Choice of technique; Fixed Capital; Rent; Markup Pricing

JEL Classification: B51; C67; D24; D33;

* I thank Andres Lazzarini for helpful comments. All errors and infelicities remain the responsibility of the author.

1. Introduction

Post-Sraffian price theory provides methods for determining the cost-minimizing technique in models of production of commodities (Kurz & Salvadori 1995). In models of circulating and pure fixed capital, the cost-minimizing technique, at a given wage or rate of profits, contributes its wage curve to the outer frontier of all wage curves. A well-known diagram illustrates this application, with an exogenous specification of the distribution of income (Bharadwaj 1963).

This article describes a different diagram, one that depicts how the analysis of the choice of technique varies with perturbations of selected parameters of the models. The abscissa is a parameter of the model. This parameter can be a relative markup in models of industries with market power. It can be time in models of structural economic dynamics (Pasinetti 1993). The ordinate is a variable specifying income distribution, such as the wage. The maximum wage and switch points are plotted as functions of the model parameter. Fluke switch points, in which the qualitative analysis of the choice of technique varies, provide vertical divisions in the diagram. The cost-minimizing technique is labeled among ranges of the distributive parameter.

A fluke switch point is a switch point in which almost all variations in model parameters destroy a qualitative property of the switch point. Examples include a switch point at which two wage curves are tangent and a switch point at which more than two wage curves intersect. Vienneau (2018, 2019, 2021, 2022, and 2024) partitions parameter spaces based on fluke switch points.

Three examples are provided of this diagram of variations in the choice of technique. The first example is of a three-commodity example of circulating capital and markup pricing (D'Agata 2018 and Zambelli 2018). This variation on an example in Vienneau (2024) exhibits the emergence of reswitching and capital-reversing, among other phenomena. The second example is of a pure fixed capital model. Following Vienneau (2021), a variation of an example from Schefold (1980) illustrates the entanglement of structural dynamics with the choice of the economic life of a machine. The third example extends the analysis of D'Agata (1983) to include markup pricing. It demonstrates the diagram even when the cost-minimizing technique is not unique away from switch points, the cost-minimizing technique does not exist, and the cost-minimizing technique cannot necessarily be found by constructing the outer frontier from wage curves.

The analysis in this article provides several novel contributions in addition to demonstrating the diagram for visualizing variations in the analysis of the choice of technique. The first example extends the critique of the Cambridge capital controversy, in general along the lines of Vienneau (2024). The second example re-iterates that longer economic lifetimes of machines have no necessary connection with increased capital intensity (Steedman 2020). Such results continue to pose a problem for the Austrian school of economics (Fratini 2019). The third example is the first analysis of a combination of markup pricing and intensive rent in post-Sraffian price theory. An appendix provides a specification of an algorithm for finding model parameters with a fluke switch point.

2. An Example with Circulating Capital and Markup Pricing

A variation of a numerical example in Vienneau (2024) provides the first example. Consider an economy which produces three commodities, iron, steel, and corn, with the technology specified in Table 1. Two processes are available for producing each commodity. The coefficients of production in a column specify the person-years of labor, tons of iron, tons of steel, and bushels of corn required to produce a unit of output of the given industry.

Table 1 – Technology for a Three-Commodity Example

Input	Iron		Steel		Corn	
	a	**b**	**c**	**d**	**e**	**f**
Labor	1/3	1/10	5/2	7/20	1	3/2
Iron	1/6	2/5	1/200	1/100	1	0
Steel	1/200	1/400	1/4	3/10	0	1/4
Corn	1/300	1/300	1/300	0	0	0

Table 2 – Technique for a Three-Commodity Example

Technique	Iron Process	Steel Process	Corn Process
Alpha	a	c	e
Beta	a	c	f
Gamma	a	d	e
Delta	a	d	f
Epsilon	b	c	e
Zeta	b	c	f
Eta	b	d	e
Theta	b	d	f

Eight techniques (Table 2) are defined for this technology. Each technique is defined by the operation of one process in the three industries. All three commodities are Sraffian basics in all techniques. That is, each commodity is a direct or indirect input in the production of all commodities. For example, iron is used directly as an input in the first corn-producing process, and steel is used indirectly in producing corn with this process since steel is an input in either iron-producing process.

Prices of production are defined here for given ratios of markups among industries. The ratios of rates of profits among industries are assumed stable, but rates of profits are not necessarily uniform. Lack of uniformity in rates of profits can result from variations in evaluations of profits among industries due to idiosyncratic properties of investment; from barriers to entry arising from, for example, secrets in manufacture; and from legal monopolies (D'Agata 2018). Let $s_1 \cdot r$, $s_2 \cdot r$, and $s_3 \cdot r$ be the rate of profits in the iron, steel, and corn industries respectively. I call r the scale factor for the rate of profits. A system of equations must be satisfied for prices of production for a given technique. For example, suppose $\mathbf{a}_{0,\alpha}$ is a three-element row vector of direct labor coefficients, $\mathbf{A}_\alpha$ is the square Leontief input matrix for the Alpha technique, and $\mathbf{B}_\alpha$ is the identity matrix. Let $\mathbf{S}$ be a diagonal matrix with the markups along the principal diagonal.

Prices of production must satisfy the system of equations in Display 1:

$$\mathbf{p}_\alpha(r) \cdot \mathbf{A}_\alpha \cdot (\mathbf{I} + r \cdot \mathbf{S}) + w_\alpha(r) \cdot \mathbf{a}_{0,\alpha} = \mathbf{p}_\alpha(r) \cdot \mathbf{B}_\alpha \qquad (1)$$

where $\mathbf{p}_\alpha(r)$ is the three-element row vector of prices and $w_\alpha(r)$ is the wage. Let $\mathbf{d}$ be a three-element column vector representing the numeraire. Suppose that a bushel of corn is the numeraire, so this vector is the third column in the 3x3 identity matrix. Display 2 specifies that the price of the numeraire is unity:

$$\mathbf{p}_\alpha(r) \cdot \mathbf{d} = 1 \qquad (2)$$

The solution to this system has a single degree of freedom, which can be expressed with wage as a function of the scale factor for the rate of profits.

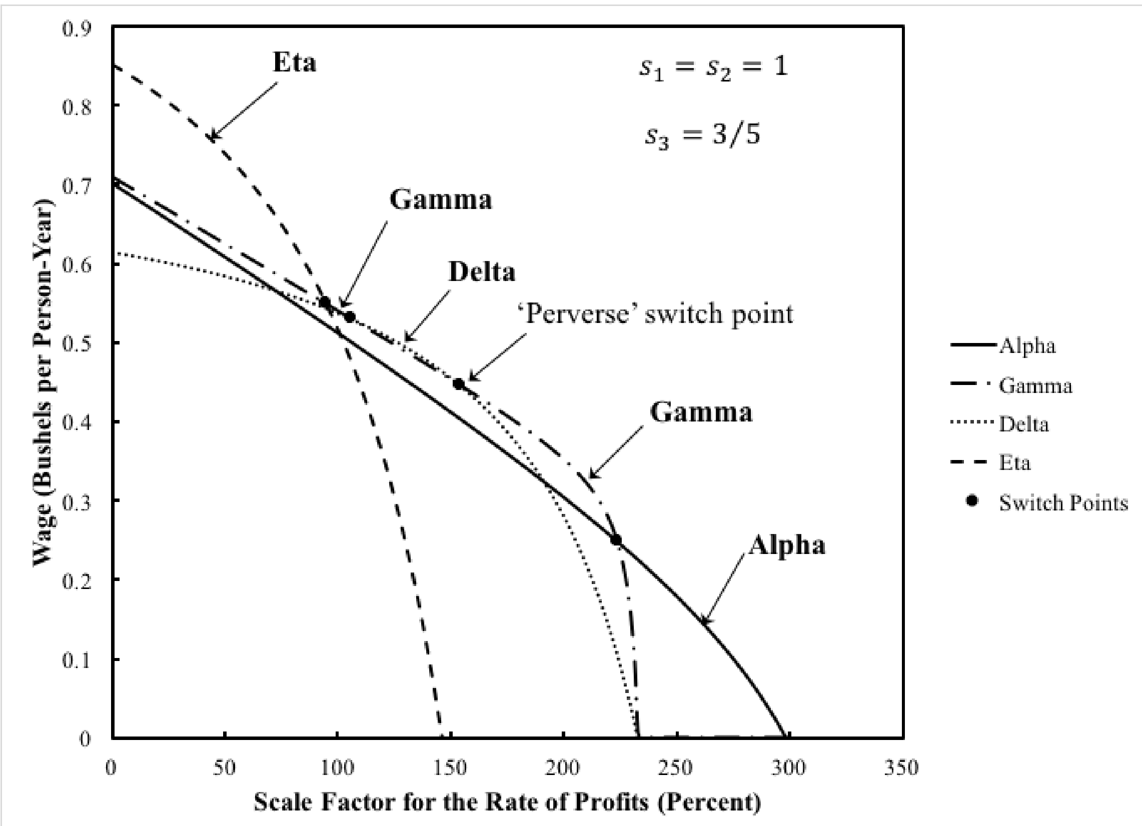

Figure 1 – The Wage Frontier for the Three-Commodity Example

Figure 1 graphs the wage curves for four techniques. The cost-minimizing technique at a given wage maximizes the scale factor for the rate of profits. The cost-minimizing technique at a given scale factor maximizes the wage. The outer frontier of all wage curves shows the variation of the cost-minimizing technique with distribution. Wage curves are graphed in Figure 1 only for the techniques on the outer frontier. This type of figure, usually for competitive markets, is the most well-known graph in post-Sraffian price theory.

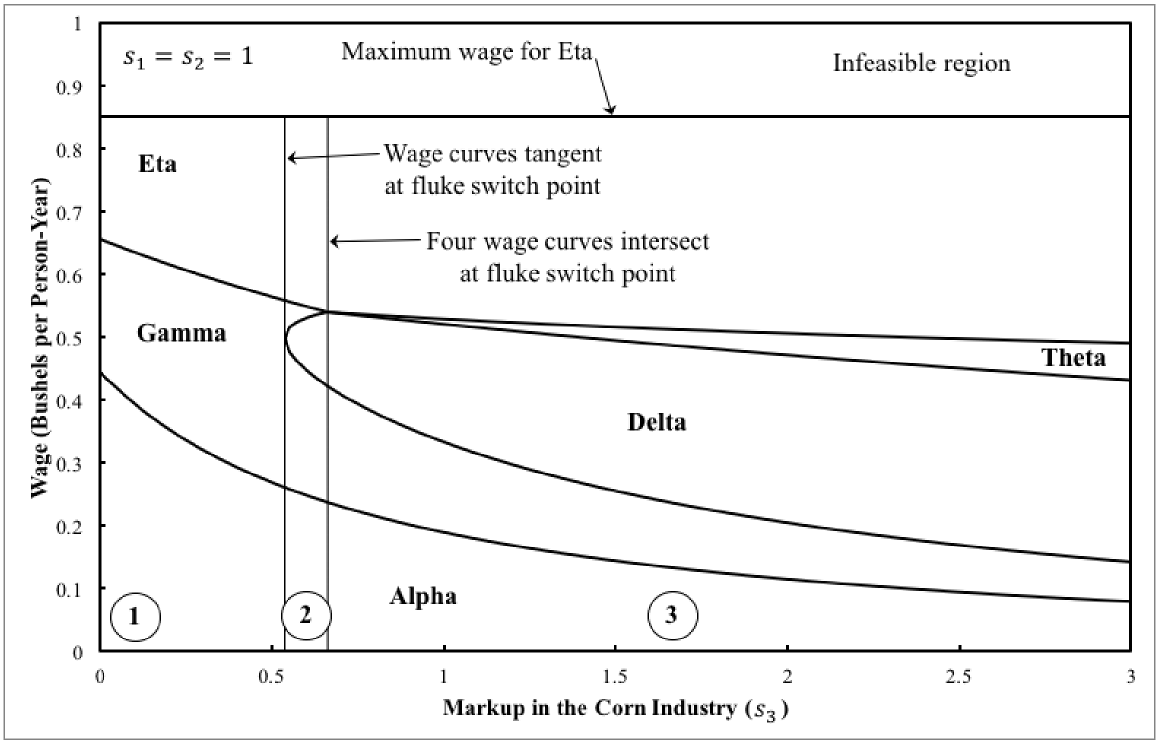

Figure 2 – Variation in the Choice of Technique for the Three-Commodity Example

Around the so-called perverse switch point, the firms in the corn industry switch from the second corn-producing process to the first at a lower wage. That is, they adopt a process that requires less labor to be hired per bushel of corn produced gross. This is known as the reverse substitution of labor (Han & Schefold 2006). For the economy as a whole, the technique adopted at a lower wage requires less labor per unit of net output. This is a consequence of capital-reversing as manifested in a comparison of stationary states (Harris 1973). Since Gamma is cost-minimizing in two discrete ranges of the wage, with Delta cost-minimizing in-between, these parameters illustrate the reswitching of techniques as well. Capital-reversing can occur without reswitching on the frontier, and the reverse substitution of labor can occur with neither reswitching nor capital-reversing occurring.

Figure 2 is the new type of diagram illustrated in this paper for depicting the analysis of the choice of technique. The abscissa is the markup in the corn industry, with given markups of unity in the iron and steel industry. The maximum wage and the wage at switch points along the frontier are plotted. The number and sequence of switch points along the wage frontier are invariant in each numbered region. Fluke switch points partition the numbered regions. Figure 3 extends Figure 2 to the right for low wages.

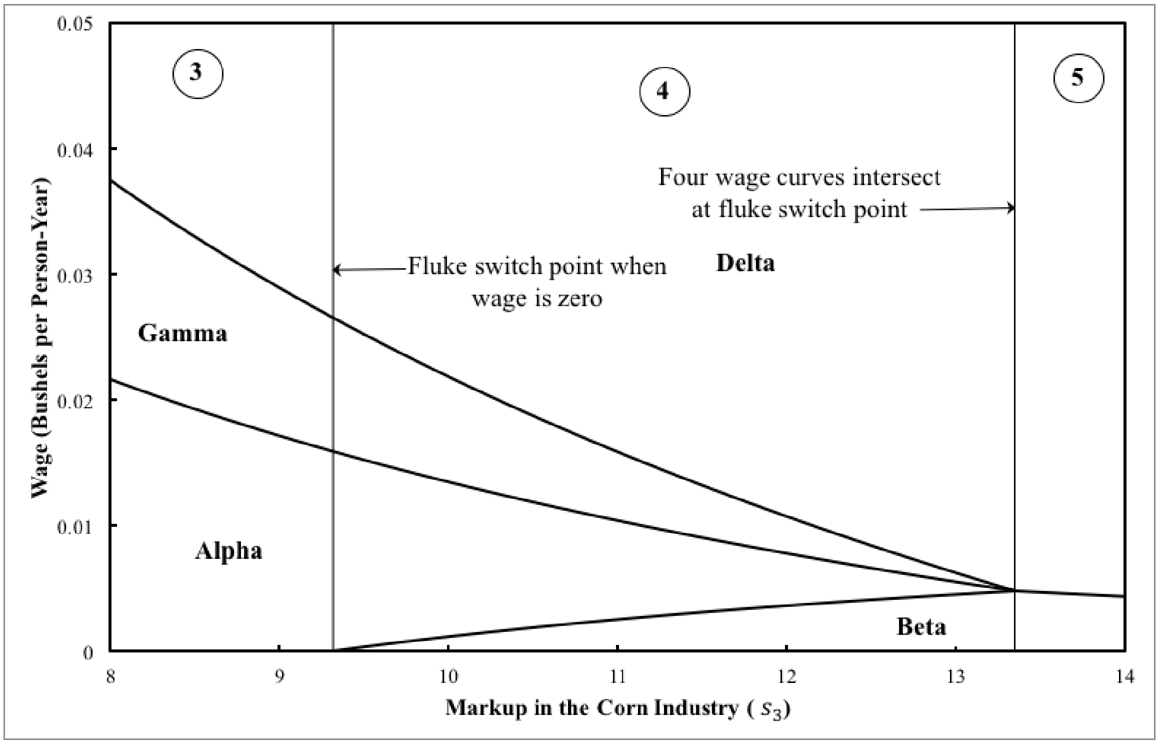

Figure 3 – Variation in the Choice of Technique (Cont'd)

The qualitative properties of the wage frontier are invariant in each numbered region in Figures 2 and 3. Table 3 describes each numbered region. The cost-minimizing technique along the wage frontier is listed, from a scale factor of the rate of profits of zero to the highest scale factor. Some salient properties of switch points and the cost-minimizing technique are summarized. Figure 1 depicts the wage frontier for a markup in the corn-industry in region 2. Region 3 includes the case in which rates of profits are uniform among industries. Reswitching does not occur in such a case, but capital-reversing and the reverse substitution of labor do.

This example allows for a graphical display showing that reswitching arises with an increased markup in corn-production, starting from a markup much less than in other industries. The 'perverse' switch point between Gamma and Delta remains on the wage frontier after the other switch point between these techniques falls off the frontier at a higher markup. Eventually, the 'perverse' switch point is no longer on the frontier when corn-production has a much higher markup than other industries.

Table 3 – Variations in the Cost-Minimizing Technique

Region	Range	Technique	Summary
1	$0 \leq r \leq r_1$	Eta	No reswitching, no capital-reversing, no reverse substitution of labor, no process recurrence.
	$r_1 \leq r \leq r_2$	Gamma	
	$r_2 \leq r \leq R_\alpha$	Alpha	
2	$0 \leq r \leq r_1$	Eta	Reswitching of techniques between Gamma and Delta, capital-reversing and the reverse substitution of labor at the switch point between Gamma and Delta at the lower wage, process recurrence of the first process in the corn industry.
	$r_1 \leq r \leq r_2$	Gamma	
	$r_2 \leq r \leq r_3$	Delta	
	$r_3 \leq r \leq r_4$	Gamma	
	$r_4 \leq r \leq R_\alpha$	Alpha	
3	$0 \leq r \leq r_1$	Eta	No reswitching. Capital-reversing and the reverse substitution of labor at the switch point between Gamma and Delta. Process recurrence of the first process in the corn industry.
	$r_1 \leq r \leq r_2$	Theta	
	$r_2 \leq r \leq r_3$	Delta	
	$r_3 \leq r \leq r_4$	Gamma	
	$r_4 \leq r \leq R_\alpha$	Alpha	
4	$0 \leq r \leq r_1$	Eta	No reswitching. Capital-reversing and the reverse substitution of labor at the switch point between Gamma and Delta. Process recurrence of both processes in the corn industry.
	$r_1 \leq r \leq r_2$	Theta	
	$r_2 \leq r \leq r_3$	Delta	
	$r_3 \leq r \leq r_4$	Gamma	
	$r_4 \leq r \leq r_5$	Alpha	
	$r_5 \leq r \leq R_\beta$	Beta	
5	$0 \leq r \leq r_1$	Eta	No reswitching, no capital-reversing, no reverse substitution of labor, no process recurrence.
	$r_1 \leq r \leq r_2$	Theta	
	$r_2 \leq r \leq r_3$	Delta	
	$r_4 \leq r \leq R_\beta$	Beta	

3. An Example with Fixed Capital and Technical Change

A variation of an example from Schefold (1980) illustrates the application of the visualization approach in this paper to a model of pure fixed capital. Table 4 presents the coefficients of production. Machines that have a physical life of two years are produced in the machine industry by workers with the use of circulating capital. The workers use the machines and circulating capital to produce a consumption good in the corn industry. The coefficients display the effects of technical progress. The inputs in the machine industry and the non-machine inputs in producing corn continuously decline, at possibly different rates in the two industries. At any given time, coefficients of production are defined, and prices of production can be calculated.

Table 4 – Technology for an Example with Fixed Capital

Input	Machine Industry	Corn Industry	
	I	II	III
Labor	$a_{0,1} = \frac{7}{25} \cdot e^{-\sigma \cdot t}$	$a_{0,2} = 3 \cdot e^{-\phi \cdot t}$	$a_{0,3} = \frac{14}{5} \cdot e^{-\phi \cdot t}$
Corn	$a_{1,1} = \frac{4}{25} \cdot e^{-\sigma \cdot t}$	$a_{1,2} = \frac{4}{25} \cdot e^{-\phi \cdot t}$	$a_{1,3} = \frac{2}{3} \cdot e^{-\phi \cdot t}$
New Machines	0	1	0
Old Machines	0	0	1
Outputs			
Corn	0	$b_{1,2} = 1$	$b_{1,3} = 1$
New Machines	1	0	0
Old Machines	0	1	0

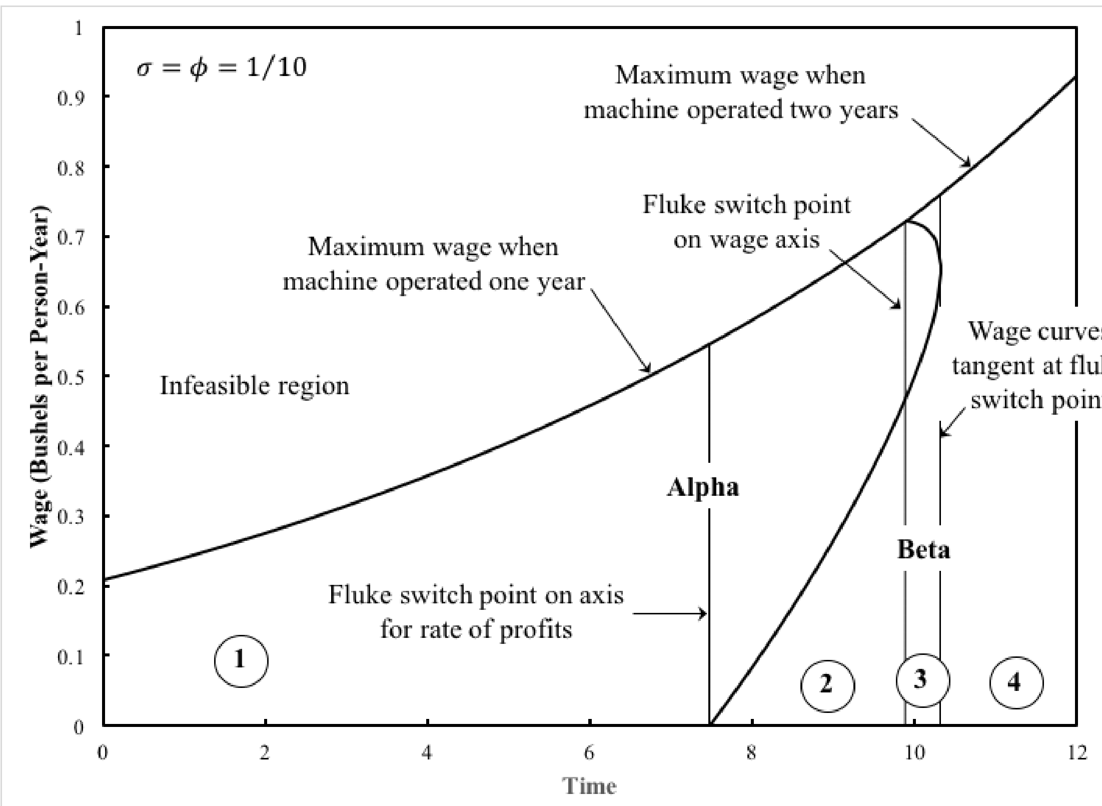

Figure 4 – Variation in the Choice of the Technique for the Example with Fixed Capital

The efficiency of the machine over its physical lifetime cannot be defined in physical terms. The person-years of labor needed to tend the machine over these two years declines, while the corn input needed per machine increases. This model of technical change is not claimed to be realistic. The point is to show how the type of diagram can depict variations in the analysis of the choice of technique when all coefficients of production depend on a single parameter, denoted here as time. Each coefficient of production could be specified as a separate function of time, with coefficients being constant if wanted. For Harrod-Neutral technical change, only the labor coefficients decrease, all at the same rate. Biases in technical change can be introduced with particular assumptions. In a model with more produced commodities, one might specify more concrete assumptions about inputs of energy, lubricants, semi-finished products, and so on.

The choice of technique corresponds here to the choice of the economic life of the machine. Assume free disposal, a key assumption that eliminates the use of this model in addressing ecological concerns. The machine is operated for one year under the Alpha technique. The machine is operated for the full physical life of two years under the Beta technique.

Prices of production are specified for the two techniques in this example too, as in the system of equations in Display 1. Alpha has 2x2 input and output matrices, while Beta has 3x3 input and output matrices. Assume competitive markets, and let a bushel of corn be the numeraire. A wage curve can be constructed for each technique, and the cost-minimizing technique is the technique with its wage curve on the outer frontier at a given wage or rate of profits. Figure 4 depicts how the analysis of the economic life of the machine varies with time. The parameters, σ and ϕ, that control the rate of change in the machine and corn industries are taken as given.

The maximum wage rises and the economic life of the machine changes with technical progress. Table 5 summarizes the choice of technique in each numbered region in the diagram. Initially, in region 1, the economic life of a machine is one year, whatever the distribution of income. Ultimately, in region 4, it is the physical life of two years, also independent of distribution. In between these times a switch point and then a reswitching example emerges.

Around the switch point in region 2 and at the lower wage in region 3, it pays to run the machine for a second year for lower wages or higher rates of profits. Around this switch point, cost-minimizing firms hire more labor in region 2, given net output, at a lower wage. In region 3, they hire less labor at a lower wage around this switch point. In either case, the adoption of a less labor-intensive technique is associated with greater net output per worker.

Table 5 – Variations in the Economic Life of a Machine

Region	Range	Technique	Summary
1	$0 \leq r \leq R_\alpha$	Machine operated for one year	No switch points.
2	$0 \leq r \leq r_1$	Machine operated for one year	The switch point exhibits negative real Wicksell effects. A smaller rate of profits is associated with a shorter economic life of a machine.
	$r_1 \leq r \leq R_\beta$	Machine operated for two years	
3	$0 \leq r \leq r_1$	Machine operated for two years	Reswitching, capital reversing, and the recurrence of truncation. The switch point at r_2 exhibits a positive real Wicksell effect. A smaller rate of profits is associated with a shorter economic life of a machine around this switch point.
	$r_1 \leq r \leq r_2$	Machine operated for one year	
	$r_2 \leq r \leq R_\beta$	Machine operated for two years	
4	$0 \leq r \leq R_\beta$	Machine operated for two years	No switch points.

The economic life of a machine cannot be mapped to the capital-intensity of a technique. Adopting a technique in which a machine is run longer is not necessarily more capital-intensive in that it does not necessarily raise net output per worker. This counter-intuitive result, at least by traditional marginalist and Austrian teaching, obtains in Region 2, for example. The non-correlation of the increased life of a machine with capital-intensity is independent of reswitching and capital-reversing.

4. An Example with Intensive Rent and Markup Pricing

Consider the coefficients of production in Table 6. This example, taken from D'Agata (1983), is for the case of intensive rent. Iron and steel can each be manufactured by labor working up an input of corn. Three processes are known for producing corn, each operating on an acre of land per bushel of corn produced and each requiring input of labor, iron, steel, and corn. Assume that one hundred acres of land are available and that the required net output consists of 90 tons of iron, 60 tons of steel, and 19 bushels of corn. The net output is also the numeraire.

Table 7 specifies which processes are operated for the six techniques in this example. The scarcity of land is seen in the possibility of two corn-producing processes operating side-by-side. Only the Alpha, Delta, and Epsilon techniques can produce the required net

Table 6 – Technology for an Example with Intensive Rent

	Iron	Steel	Corn		
	I	**II**	**III**	**IV**	**V**
Labor	1	1	1	11/5	1
Land	0	0	1	1	1
Iron	0	0	1/10	1/10	1/10
Steel	0	0	2/5	1/10	1/10
Corn	1/10	3/5	1/10	3/10	2/5

Table 7 – Technique for an Example with Intensive Rent

Technique	Iron Process	Steel Process	Corn Process(es)
Alpha	I	II	III
Beta	I	II	IV
Gamma	I	II	V
Delta	I	II	III, IV
Epsilon	I	II	III, V
Zeta	I	II	IV, V

output for this example. The infeasible techniques are ignored in the remainder of this analysis.

Prices of production for a technique in which two processes are operated to produce corn include the payment of rent to landlords. Consider the Delta technique. Let the rate of profits be $s_1 \cdot r$, $s_2 \cdot r$, and $s_3 \cdot r$ in the iron, steel, and corn industries, respectively. Let ϱ be rent per acre. Prices of production satisfy the equations in Displays 3, 4, 5, and 6:

$$\frac{1}{10} \cdot p_3^{\delta}(r) \cdot (1 + s_1 \cdot r) + w_{\delta}(r) = p_1^{\delta}(r) \tag{3}$$

$$\frac{3}{5} \cdot p_3^{\delta}(r) \cdot (1 + s_2 \cdot r) + w_{\delta}(r) = p_2^{\delta}(r) \tag{4}$$

$$\begin{aligned}\left[\frac{1}{10} \cdot p_1^{\delta}(r) + \frac{2}{5} \cdot p_2^{\delta}(r) + \frac{1}{10} \cdot p_3^{\delta}(r)\right] \cdot (1 + s_3 \cdot r)& \\ + \varrho + w_{\delta}(r) = p_3^{\delta}(r)&\end{aligned} \tag{5}$$

$$\begin{aligned}\left[\frac{1}{10} \cdot p_1^{\delta}(r) + \frac{1}{10} \cdot p_2^{\delta}(r) + \frac{3}{10} \cdot p_3^{\delta}(r)\right] \cdot (1 + s_3 \cdot r)& \\ + \varrho + \frac{11}{5} \cdot w_{\delta}(r) = p_3^{\delta}(r)&\end{aligned} \tag{6}$$

The specification of the numeraire imposes the constraint in Display 7:

$$90 \cdot p_1^{\delta}(r) + 60 \cdot p_2^{\delta}(r) + 19 \cdot p_3^{\delta}(r) = 1 \tag{7}$$

This is a system of five equations for six variables, that is, the three prices, the wage, rent per acre, and the scale factor for the rate of profits.

Rent can be eliminated from Equations 5 and 6, leaving a system of equations of the form in Display 1. Intensive rent arises in models of joint production, and wage curves and the analysis of the choice of technique do not necessarily have properties that they do in models of circulating capital. Wage curves can slope up with some numeraires, including when the corresponding technique is cost-minimizing (Woods 1990). The cost-minimizing technique is not found by constructing the outer frontier of wage curves (Bidard & Klimovsky 2004). The cost-minimizing technique may not be unique away from switch points, and a cost-minimizing technique may not exist.

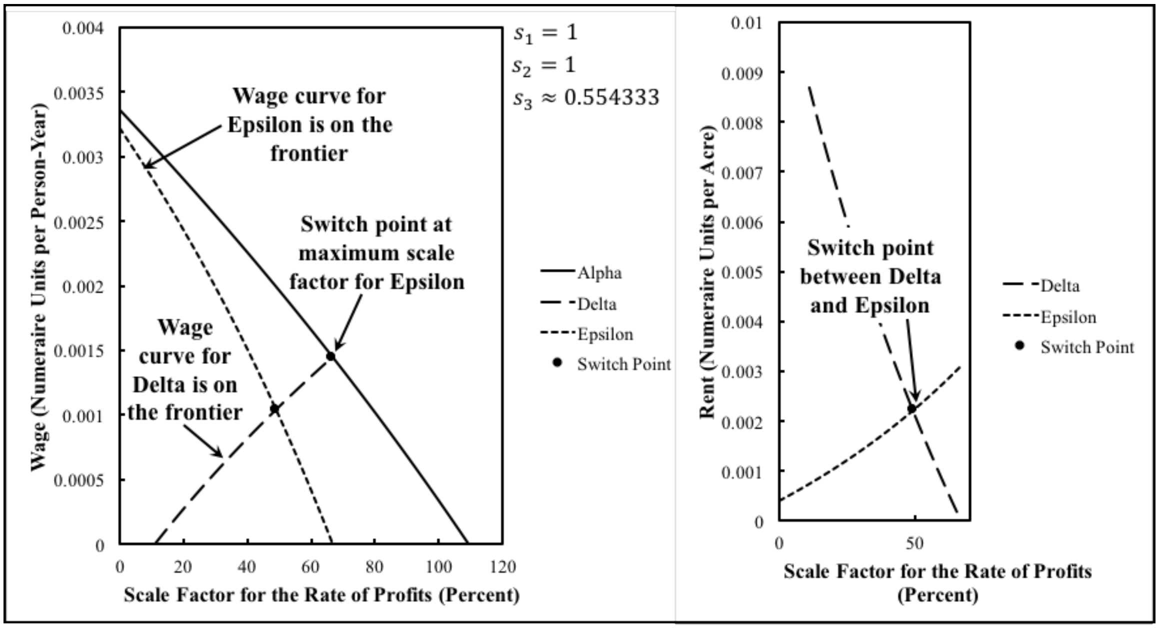

Figure 5 – Wage and Rent Curves for Fluke Case

Figure 5 depicts the wage and rent curves for the feasible techniques for a fluke case. The wage curves for Alpha and Delta intersect at a scale factor for the rate of profits that is the maximum possible for the Epsilon technique. This fluke case is associated with a qualitative change in the range of the scale factor for the rate of profits in which no cost-minimizing technique exists. The wage frontier consists of the wage curves for the Delta and Epsilon techniques up to the switch point between them. The wage frontier ends there. No technique is cost-minimizing for a scale factor between this switch point and the maximum scale factor for the rate of profits for Alpha.

In the range of the scale factor for the range of profits from zero to the scale factor at which the wage for Delta is zero, only the Alpha and Epsilon techniques have wage curves that are eligible to lie on the wage frontier; The wage curve for Delta lies below the axis for the scale factor for the rate of profits. All three wage curves are eligible for a scale factor between the minimum for Delta and the scale factor at which the wage curves for Alpha and Delta intersect. Between the scale factor at this intersection and the maximum for Alpha, the rent curve for Delta and the wage curve for Epsilon lie below the axis for the scale factor.

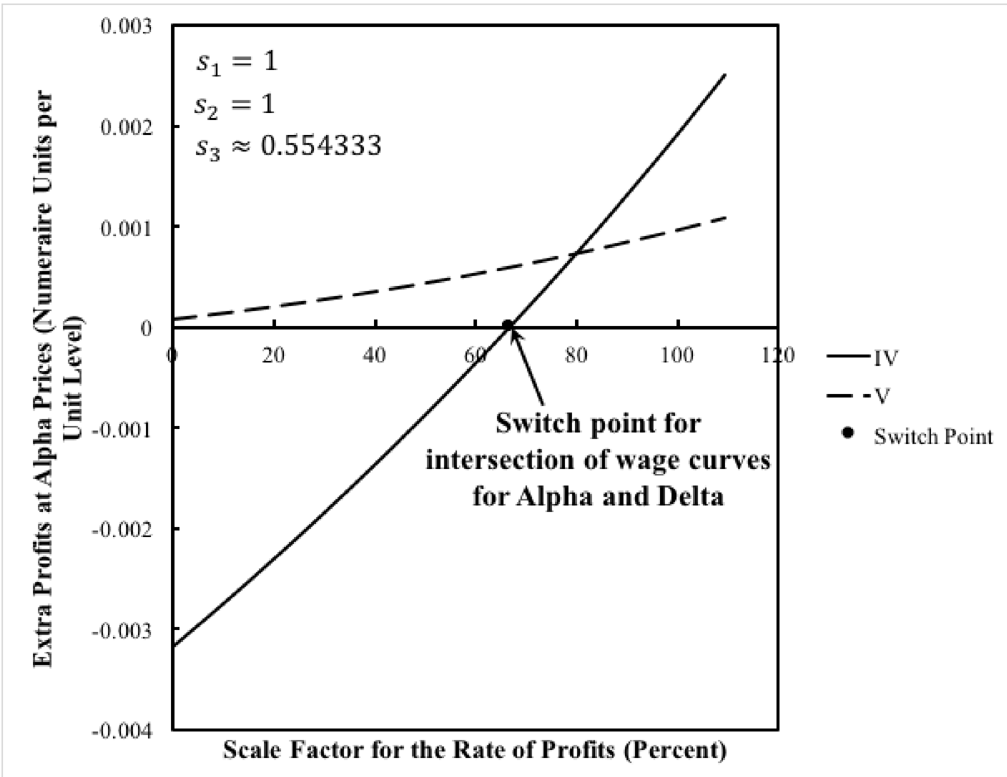

Figure 6 – Extra Profits at Alpha Prices

One can plot extra profits for each process to determine if a technique is cost-minimizing at a given scale factor for the rate profits. Figure 6 graphs extra profits at Alpha prices for the two corn-producing processes not operated in Alpha. The last corn-producing process can always pay extra profits for any scale factor not exceeding the maximum scale factor, while the penultimate process can pay extra profits for any scale factor greater than that at the intersection of the Alpha and Delta wage curves. The Alpha technique is never cost-minimizing.

Figure 7 graphs extra profits for the Delta and Epsilon techniques for the process not operated in the respective technique. If the Delta technique were in operation at a scale factor greater than at the switch point between Delta and Epsilon, farmers would start to operate the fifth process, moving away from the Delta technique. If the Epsilon technique

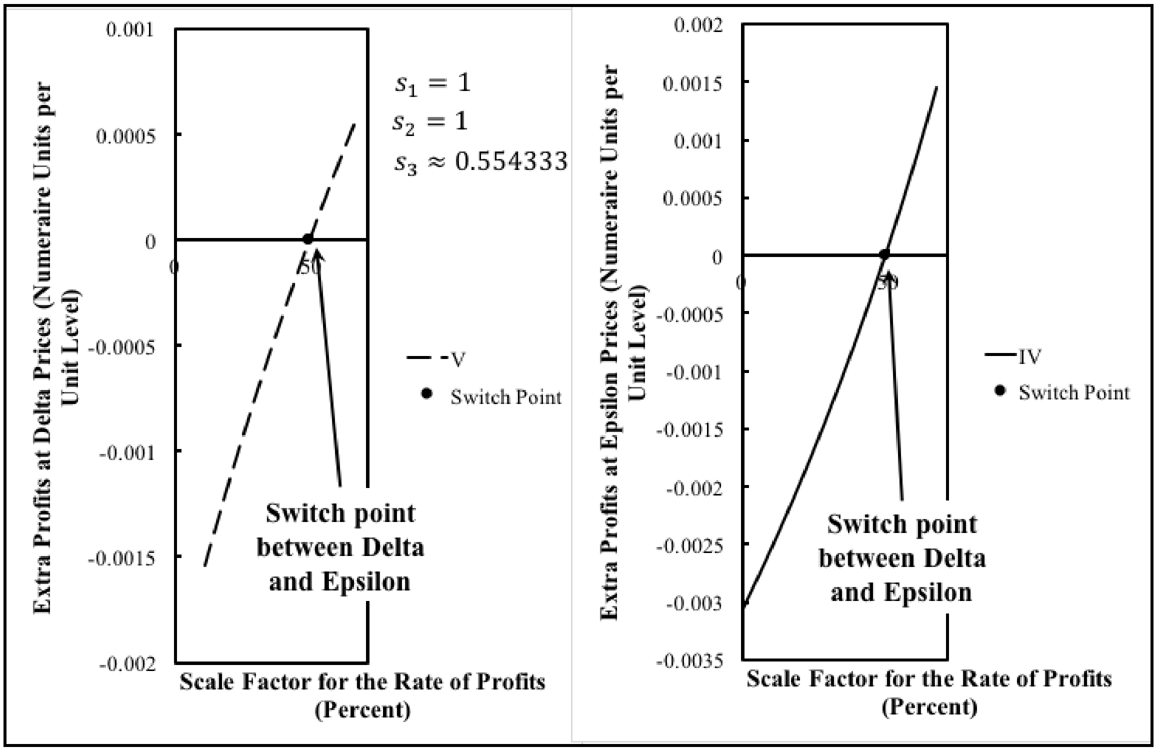

Figure 7 – Extra Profits at Delta and Epsilon Prices

were in operation in this range, farmers would start to operate the fourth process. A market algorithm (Bidard 2004) would not converge to any technique for a scale factor for the rate of profits greater than at the switch point between Delta and Epsilon and not exceeding the maximum scale factor for the Alpha technique.

Figure 8 shows the variation in the analysis of the cost-minimizing technique with perturbations of the markup up in agriculture. In drawing this figure, markups in iron and steel production, s_1 and s_2, are assumed to be unity. At the intersection between the Alpha and Delta wage curves, the rent for Delta is zero. The scale factor at this switch point is the maximum for the Delta technique. In regions 1 and 2, the maximum scale factor for Epsilon is the scale factor for which the wage turns negative. At a switch point between Alpha and Epsilon in regions 3, 4, 5, and 6, the rent for Epsilon is zero. The scale factor at such a switch point is the maximum scale factor for Epsilon in these regions.

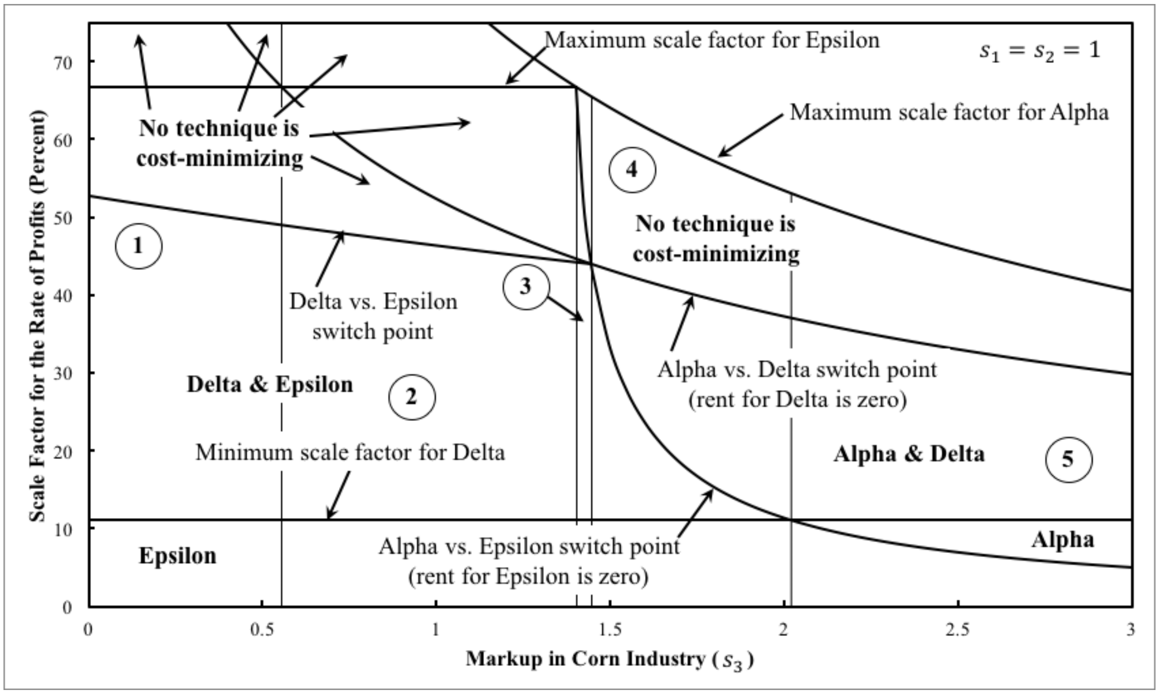

Figure 8 – Variation in the Choice of Technique for Example with Intensive Rent

The fluke cases partitioning regions 1 and 2 and regions 2 and 3 change some characteristics of the range of the scale factor of the rate of profits in which no cost-minimizing technique exists. Figure 5 illustrates the analysis of the fluke case dividing regions 1 and 2. At the fluke case partitioning regions 2 and 3, the wage curves for Alpha and Epsilon intersect at the maximum scale factor for Alpha. These two fluke cases arise, in some sense, for switch points off the frontier.

Table 8 – Variations in an Example of Intensive Rent

Region	Range	Technique	Notes
1	$0 \leq r \leq R_\delta$	Epsilon	Alpha and Epsilon have positive wage, Epsilon has positive rent.
	$R_\delta \leq r \leq r_1$	Delta and Epsilon	Alpha, Delta, and Epsilon have positive wage. Delta and Epsilon have positive rent.
	$r_1 \leq r \leq R_\varepsilon$	None	
	$R_\varepsilon \leq r \leq \hat{R}_\delta$	None	Alpha and Delta have positive wage. Delta has positive rent.
	$\hat{R}_\delta \leq r \leq R_\alpha$	None	Alpha has positive wage.

2	$0 \leq r \leq R_\delta$	Epsilon	Alpha and Epsilon have positive wage, Epsilon has positive rent.
	$R_\delta \leq r \leq r_1$	Delta and Epsilon	Alpha, Delta, and Epsilon have positive wage. Delta and Epsilon have positive rent.
	$r_1 \leq r \leq \hat{R}_\delta$	None	
	$\hat{R}_\delta \leq r \leq R_\varepsilon$	None	Alpha and Epsilon have positive wage. Epsilon has positive rent.
	$R_\varepsilon \leq r \leq R_\alpha$	None	Alpha has positive wage.
3	$0 \leq r \leq R_\delta$	Epsilon	Alpha and Epsilon have positive wage, Epsilon has positive rent.
	$R_\delta \leq r \leq r_1$	Delta and Epsilon	Alpha, Delta, and Epsilon have positive wage. Delta and Epsilon have positive rent.
	$r_1 \leq r \leq \hat{R}_\delta$	None	
	$\hat{R}_\delta \leq r \leq \hat{R}_\epsilon$	None	Alpha and Epsilon have positive wage. Epsilon has positive rent.
	$\hat{R}_\epsilon \leq r \leq R_\alpha$	None	Alpha has positive wage.
4	$0 \leq r \leq R_\delta$	Epsilon	Alpha and Epsilon have positive wage, Epsilon has positive rent.
	$R_\delta \leq r \leq \hat{R}_\epsilon$	Delta and Epsilon	Alpha, Delta, and Epsilon have positive wage. Delta and Epsilon have positive rent.
	$\hat{R}_\epsilon \leq r \leq \hat{R}_\delta$	Alpha and Delta	Alpha and Delta have positive wage. Delta has positive rent.
	$\hat{R}_\delta \leq r \leq R_\alpha$	None	Alpha has positive wage.
5	$0 \leq r \leq \hat{R}_\epsilon$	Epsilon	Alpha and Epsilon have positive wage, Epsilon has positive rent.
	$\hat{R}_\epsilon \leq r \leq R_\delta$	Alpha	Alpha has positive wage.
	$R_\delta \leq r \leq \hat{R}_\delta$	Alpha and Delta	Alpha and Delta have positive wage. Delta has positive rent.
	$\hat{R}_\delta \leq r \leq R_\alpha$	None	Alpha has positive wage.
6	$0 \leq r \leq \hat{R}_\epsilon$	Epsilon	Alpha and Epsilon have positive wage, Epsilon has positive rent.
	$\hat{R}_\epsilon \leq r \leq R_\alpha$	Alpha	Alpha has positive wage.

The three remaining fluke switch points, of which only two are shown, are associated with a change in which techniques are cost-minimizing at different ranges of the scale factor for the rate of profits. The fluke case partitioning regions 3 and 4 is one in which the wage curves for Alpha, Delta, and Epsilon all intersect at a single switch point. The fluke case partitioning regions 4 and 5 is one where the wage curves for Alpha and Epsilon intersect at the scale factor where the wage for Delta first turns positive. A fluke case exists off to the right where the wage curves for Alpha and Delta intersect at the maximum scale factor for the rate of profits for Alpha. At that switch point, Delta has a scale factor for the rate of profits of zero percent and a rent of zero. Tables 8 and 9 show how the analysis of the choice of technique varies among the numbered regions. As in the previous examples, one can relate the variation in the analysis of the choice of technique to the fluke cases.

5. Conclusion

The examples demonstrate the flexibility of the diagram for visualizing the analysis of the choice of technique. The first example illustrates its application to one markup among several, where markups are parameters in a model of circulating capital. The second example is one of structural economic dynamics in a pure fixed capital model. All of the input coefficients of production, other than the machine input, vary parametrically with time. The third example demonstrates that these visualization techniques and perturbation analysis can be applied to an example where the cost-minimizing technique is not found from a frontier of wage curves.

Appendix: How to Find Fluke Cases

This appendix describes how to find a fluke switch point. To illustrate the approach, assume $s_1 = s_2 = 1$ in the example in Section 2. Consider locating s_3, the markup in the corn industry, such that the wage curves for Gamma, Delta, Eta, and Theta intersect at a single switch point. One wants to find a function one of whose zeros is the desired markup.

Given the markups s_1, s_2, and s_3, the wage and prices under Gamma are rational functions of the scale factor for the rate of profits:

$$w_\gamma(r) = \frac{f_3^\gamma \cdot r^3 + f_2^\gamma \cdot r^2 + f_1^\gamma \cdot r + f_0^\gamma}{g_2^\gamma \cdot r^2 + g_1^\gamma \cdot r + g_0^\gamma} \tag{A-1}$$

$$p_1^\gamma(r) = \frac{u_2^\gamma \cdot r^2 + u_1^\gamma \cdot r + u_0^\gamma}{g_2^\gamma \cdot r^2 + g_1^\gamma \cdot r + g_0^\gamma} \tag{A-2}$$

$$p_2^\gamma(r) = \frac{v_2^\gamma \cdot r^2 + v_1^\gamma \cdot r + v_0^\gamma}{g_2^\gamma \cdot r^2 + g_1^\gamma \cdot r + g_0^\gamma} \tag{A-3}$$

Recall that the price of corn is unity. The coefficients of the polynomials are functions of the coefficients of production and the markups.

The Delta technique differs from Gamma in the process for producing corn. Display A-4 defines the extra profits obtained in operating the second corn-producing process at Gamma prices:

$$\frac{h_1(r)}{g_2^\gamma \cdot r^2 + g_1^\gamma \cdot r + g_0^\gamma} = 1 - \left[\left(a_{1,3}^f \cdot p_1^\gamma(r) + a_{2,3}^f \cdot p_2^\gamma(r) + a_{3,3}^f\right) \cdot (1 + s_3 \cdot r) + w_\gamma(r) \cdot a_{0,3}^f\right] \quad \text{(A-4)}$$

A switch point between Gamma and Delta is found as an appropriate zero of $h_1(r)$, which is a cubic polynomial. Denote $r_1(s_3)$ as the zero sought for the fluke case.

Display A-5 defines extra profits obtained in operating the second iron-producing process at Gamma prices:

$$\frac{h_2(r)}{g_2^\gamma \cdot r^2 + g_1^\gamma \cdot r + g_0^\gamma} = p_1^\gamma(r) - \left[\left(a_{1,1}^b \cdot p_1^\gamma(r) + a_{2,1}^b \cdot p_2^\gamma(r) + a_{3,1}^b\right) \cdot (1 + s_1 \cdot r) + w_\gamma(r) \cdot a_{0,1}^b\right] \quad \text{(A-5)}$$

An appropriate zero of $h_2(r)$ is a switch point between Gamma and Eta. Denote $r_2(s_3)$ as the zero sought for the fluke case.

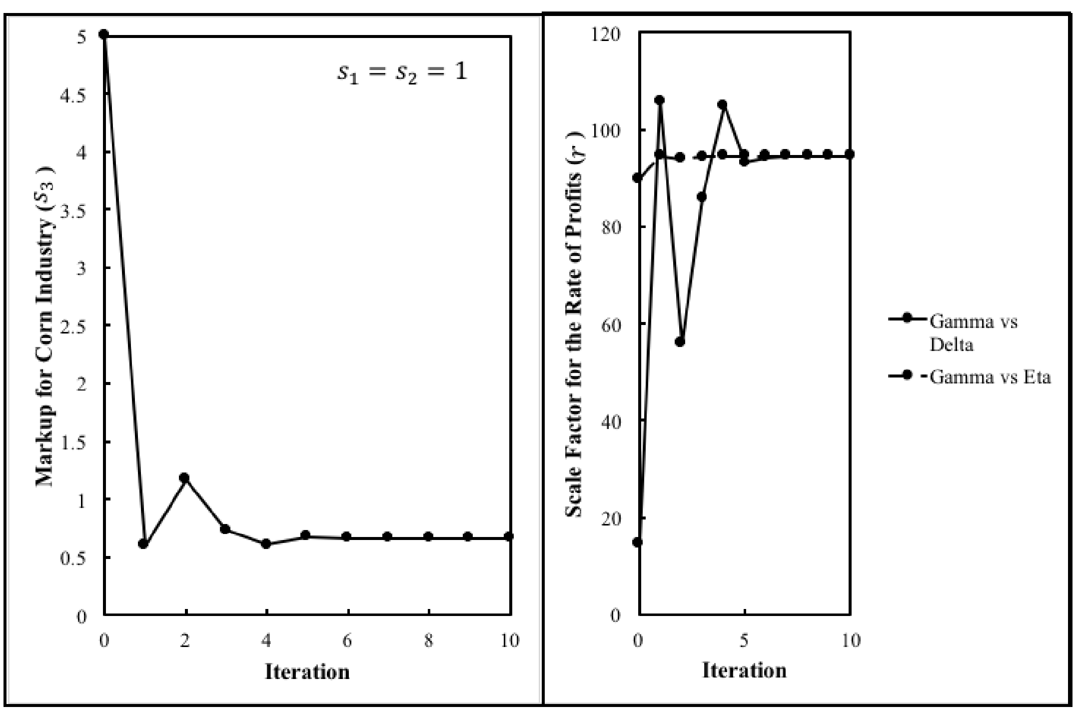

Figure A-1 – Convergence on Fluke Case

Consider the function defined in A-6:

$$h(s_3) = r_2(s_3) - r_1(s_3) \tag{A-6}$$

A zero of $h(s_3)$ is such that the wage curves for Gamma, Delta, Eta, and Theta intersect at a single switch point. At a switch point for Gamma, Delta, and Eta, neither extra profits nor extra costs will be obtained in operating either iron-producing or corn-producing processes. Since the same steel-producing process is operated in all four techniques, Theta is also cost-minimizing at this switch point.

One can find such a zero by applying Newton's method to two initial guesses, as illustrated in Figure A-1. Some experimentation allows one to determine two initial guesses, s_3^0 and s_3^1, for the markup in the corn industry and which roots of the cubic polynomials are wanted. Display A-7 specifies the slope of a linear approximation to the function whose zero is sought:

$$m_{i+2} = \frac{h(s_3^i) - h(s_3^{i+1})}{s_3^i - s_3^{i+1}},\ i = 0,1,2, \ldots \tag{A-7}$$

Display A-8 specifies the intercept with the ordinate:

$$b_{i+2} = h\left(s_3^{i+1}\right) - m_{i+2} \cdot s_3^{i+1},\ i = 0,1,2, \ldots \tag{A-8}$$

Display A-9 specifies the next iteration.

$$s_3^{i+2} = -\,b_{i+2}/m_{i+2},\ i = 0,1,2, \ldots \tag{A-9}$$

In my experience, Newton's method converges fairly rapidly in this application of finding fluke switch points.

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